Uniqueness Theorem

Consider the symmetric, time-harmonic form of Maxwell's equations given by

$$\nabla \times \boldsymbol{E} = -\boldsymbol{M}^t \tag{1a}$$
$$\boldsymbol{M}^t = j\omega \boldsymbol{B} + \boldsymbol{M}^i \tag{1b}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^t \tag{1c}$$
$$\boldsymbol{J}^t = j\omega \boldsymbol{D} + \boldsymbol{J}_c + \boldsymbol{J}^i \tag{1d}$$

where the superscript t refers to a *total* current density. For example, the electric current density $\boldsymbol{J}^t$ is the sum of the displacement current density $j\omega\boldsymbol{D}$, the conduction current density $\boldsymbol{J}_c = \sigma\boldsymbol{E}$, and the impressed current $\boldsymbol{J}^i$.

Poynting's Theorem requires

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}^*) + \boldsymbol{E} \cdot \boldsymbol{J}^{t*} + \boldsymbol{H}^* \cdot \boldsymbol{M}^t = 0. \tag{2}$$

The integral form of this theorem is found using the divergence theorem,

$$\oint_S (\boldsymbol{E} \times \boldsymbol{H}^*) \cdot d\boldsymbol{s}' + \iiint_V (\boldsymbol{E} \cdot \boldsymbol{J}^{t*} + \boldsymbol{H}^* \cdot \boldsymbol{M}^t) dv' = 0 \tag{3}$$

Now consider a set of sources $\boldsymbol{J}$ and $\boldsymbol{M}$ acting in a linear medium, bound by a closed surface S, as shown in Figure 1.

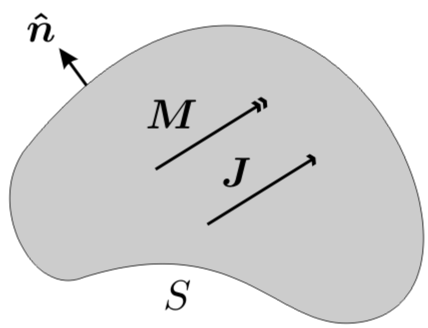

Figure 1: Physical problem

Fields within S must satisfy

$$-\nabla \times \boldsymbol{E} = j\omega\dot{\mu}\boldsymbol{H} + \boldsymbol{M}^i \tag{4a}$$
$$\begin{aligned}\nabla \times \boldsymbol{H} &= j\omega\dot{\epsilon}\boldsymbol{E} + \boldsymbol{J}_c + \boldsymbol{J}^i \\ &= (\sigma + j\omega\dot{\epsilon})\boldsymbol{E} + \boldsymbol{J}^i\end{aligned} \tag{4b}$$

where $\dot{\epsilon}$ and $\dot{\mu}$ are the complex-valued permittivity and permeability of the medium whose imaginary part contributes to loss in the medium.

We aim to show that there is only a single (i.e. unique) solution to Maxwell's equations. To prove this, suppose that there are actually two solutions to these equations: $\boldsymbol{E}^a, \boldsymbol{H}^a$ and $\boldsymbol{E}^b, \boldsymbol{H}^b$. Now consider the differences

$$\Delta \boldsymbol{E} = \boldsymbol{E}^a - \boldsymbol{E}^b \tag{5a}$$
$$\Delta \boldsymbol{H} = \boldsymbol{H}^a - \boldsymbol{H}^b. \tag{5b}$$

If we substitute these solutions into (4a), we obtain

$$-\boldsymbol{\nabla} \times \boldsymbol{E}^a = j\omega\dot{\mu}\boldsymbol{H}^a + \boldsymbol{M}^i \tag{6a}$$
$$-\boldsymbol{\nabla} \times \boldsymbol{E}^b = j\omega\dot{\mu}\boldsymbol{H}^b + \boldsymbol{M}^i. \tag{6b}$$

Subtracting these equations, within S we have

$$-\boldsymbol{\nabla} \times (\Delta \boldsymbol{E}) = j\omega\dot{\mu}(\Delta \boldsymbol{H}). \tag{7}$$

Similarly,

$$\boldsymbol{\nabla} \times (\Delta \boldsymbol{H}) = (\sigma + j\omega\dot{\epsilon})(\Delta \boldsymbol{E}) \tag{8}$$

within S. Equations (7) and (8) establish that the difference fields $\Delta \boldsymbol{E}$ and $\Delta \boldsymbol{H}$ obey the form Maxwell's equations in the *source-free region*. Therefore, any electric / magnetic currents used in the analysis of the difference fields should not involve impressed current terms.

The condition for a unique solution to (4) would obviously be

$$\Delta \boldsymbol{E} = \Delta \boldsymbol{H} = 0; \tag{9}$$

that is, $\boldsymbol{E}^a = \boldsymbol{E}^b$ and $\boldsymbol{H}^a = \boldsymbol{H}^b$. Let's apply Poynting's Theorem (3) to the difference fields $\Delta \boldsymbol{E}$ and $\Delta \boldsymbol{H}$. Given that they behave as if they were in a source-free region, the difference currents only contain non-impressed terms, i.e.,

$$\Delta \boldsymbol{J}^t \to j\omega\dot{\epsilon}\Delta \boldsymbol{E} + \sigma\Delta \boldsymbol{E} \tag{10a}$$
$$\Delta \boldsymbol{M}^t \to j\omega\dot{\mu}\Delta \boldsymbol{H}. \tag{10b}$$

Therefore,

$$\oint_S (\Delta \boldsymbol{E} \times \Delta \boldsymbol{H}^*) \cdot d\boldsymbol{s}' + \iiint_V \left[\Delta \boldsymbol{E} \cdot (-j\omega\dot{\epsilon}^*\Delta \boldsymbol{E}^* + \sigma\Delta \boldsymbol{E}^*) + \Delta \boldsymbol{H}^* \cdot (j\omega\dot{\mu}\Delta \boldsymbol{H})\right] dv' = 0, \tag{11}$$

which reduces to

$$\oint_S (\Delta \boldsymbol{E} \times \Delta \boldsymbol{H}^*) \cdot d\boldsymbol{s}' + \iiint_V \left[(\sigma + j\omega\dot{\epsilon}^*)|\Delta \boldsymbol{E}|^2 + j\omega\dot{\mu}|\Delta \boldsymbol{H}|^2\right] dv' = 0. \tag{12}$$

Consider the situation when

$$\oint_S (\Delta \boldsymbol{E} \times \Delta \boldsymbol{H}^*) \cdot d\boldsymbol{s}' = 0, \tag{13}$$

which then requires

$$\iiint_V \left[(\sigma + j\omega\dot{\epsilon}^*)|\Delta\boldsymbol{E}|^2 + j\omega\dot{\mu}|\Delta\boldsymbol{H}|^2\right] dv' = 0 \tag{14}$$

to satisfy Poynting's Theorem. If we consider complex material parameters in dissipative media

$$\dot{\epsilon} = \epsilon' - j\epsilon'', \quad \epsilon' > 0, \epsilon'' > 0 \tag{15a}$$

$$\dot{\mu} = \mu - j\mu, \quad \mu' > 0, \mu'' > 0 \tag{15b}$$

then

$$\iiint_V \left[(\sigma + j\omega\epsilon'' - j\omega\epsilon')|\Delta\boldsymbol{E}|^2 + (\omega\mu'' + j\omega\mu')|\Delta\boldsymbol{H}|^2\right] dv' = 0. \tag{16}$$

The real and imaginary parts of this equation must be equal to zero; that is,

$$\iiint_V \left[(\sigma + j\omega\epsilon'')|\Delta\boldsymbol{E}|^2 + \omega\mu''|\Delta\boldsymbol{H}|^2\right] dv' = 0 \tag{17a}$$

$$\iiint_V \left[-j\omega\epsilon'|\Delta\boldsymbol{E}|^2 + j\omega\mu'|\Delta\boldsymbol{H}|^2\right] dv' = 0. \tag{17b}$$

Examining these equations, especially (17a), we see that in dissipative media it is impossible to satisfy these relations unless we have $\Delta\boldsymbol{E} = \Delta\boldsymbol{H} = 0$. In fact, even in lossless media, we can take the limit of as the dissipation goes to zero and this still remains true. We have therefore established uniqueness, subject to our assumption (13) which is also satisfied when $\Delta\boldsymbol{E} = \Delta\boldsymbol{H} = 0$.

Equation (13) has some additional meaning. There is a vector identity that says that

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}). \tag{18}$$

Applying this to (13) with $d\boldsymbol{s}' = \hat{\boldsymbol{n}} ds$,

$$\oint_S \Delta\boldsymbol{H}^* \cdot (\hat{\boldsymbol{n}} \times \Delta\boldsymbol{E}) ds = -\oint_S \Delta\boldsymbol{E} \cdot (\hat{\boldsymbol{n}} \times \Delta\boldsymbol{H}^*) ds = 0. \tag{19}$$

We can interpret this as follows:

- $\hat{\boldsymbol{n}} \times \Delta\boldsymbol{E} = 0$ implies that that the tangential components of $\boldsymbol{E}^a$ and $\boldsymbol{E}^b$ are identical over S; that is, the tangential components of these fields are uniquely defined.
- The same applies for the fields $\boldsymbol{H}^a$ and $\boldsymbol{H}^b$.

This in turn allows (13) to be satisfied if:

1. $\hat{\boldsymbol{n}} \times \boldsymbol{E}$ is uniquely defined on S, for then $\hat{\boldsymbol{n}} \times \Delta\boldsymbol{E} = 0$ over S;
2. $\hat{\boldsymbol{n}} \times \boldsymbol{H}$ is uniquely defined on S, for then $\hat{\boldsymbol{n}} \times \Delta\boldsymbol{H} = 0$ over S;
3. $\hat{\boldsymbol{n}} \times \boldsymbol{E}$ is uniquely defined over part of S, and $\hat{\boldsymbol{n}} \times \boldsymbol{H}$ is uniquely defined over the remainder of S.

We can verbally summarize the unique theorem as: "A field in a lossy medium is completely defined by sources within the region plus the tangential components of $\boldsymbol{E}$ over the boundary, or the tangential components of $\boldsymbol{H}$ over the boundary, or the former over part of the boundary and the latter over the rest of the boundary."