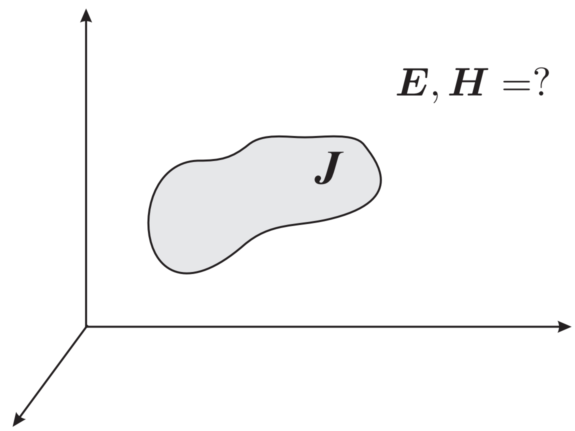
Vector Magnetic Potential

In radiation problems, the goal is to determine the radiated fields (electric and magnetic) from an antennas, knowing what currents are flowing on the antenna.

$$\boldsymbol{J} \Rightarrow \boldsymbol{E}, \boldsymbol{H} = ? \tag{1}$$



This is quite straightforward with the right tools, one of which is known as *vector potential*. We are going to make use of a vector potential to help us solve radiation problems in the near future. It is a very useful tool, although a valid question would be why not solve Maxwell's equations directly? A valid starting point might be

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + j\omega \boldsymbol{D}. \tag{2}$$

But clearly, this could be quite tricky since solving curl equations analytically is difficult even if the currents on the right hand side are known. Since a curl operation is involved, even if the currents are directed in a single direction (e.g. $\hat{\boldsymbol{z}}$), $\boldsymbol{H}$ will not be so simple (it will involve $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ components at the very least). Add to that that the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in Maxwell's equations are coupled, and the direct solution becomes quite difficult. We will use the concept of *potentials* to make our life easier.

We seek to find the radiated fields $\boldsymbol{E}, \boldsymbol{H}$ knowing the surface currents $\boldsymbol{J}$ on the antenna. In principle, we can integrate Maxwell's equation directly, as discussed above, to solve for these fields, but the solution is difficult (Figure 1. Instead, we use potentials as intermediate quantities, which we will see leads to much easier integrations, with the tradeoff being that we need to differentiate the potentials to obtain the desired fields at the end (also straightoforward).

We notice that the *solenoidal* nature of the magnetic fields from one of Maxwell's divergence relations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{3}$$

This is true because $\boldsymbol{B}$ possesses *only* a circulation: as such, its divergence must be zero since the divergence of a curl of a field is always identically zero ($\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{M} = 0$). Let us define

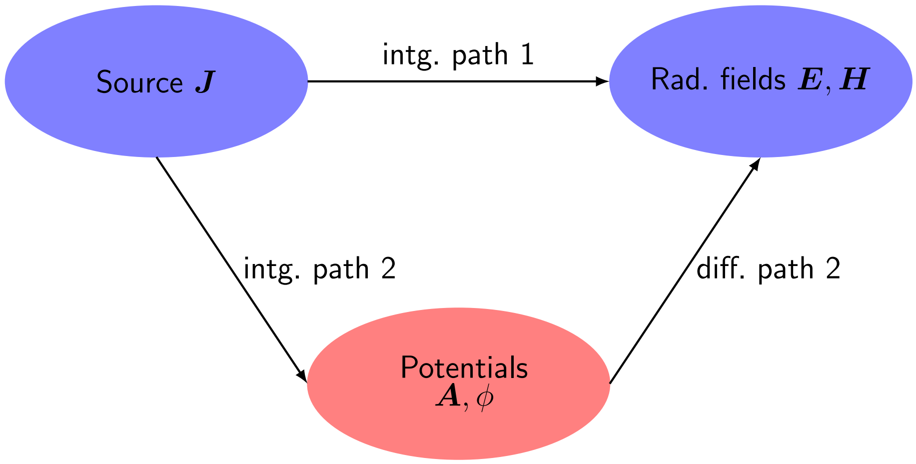


Figure 1: Solution paths for Maxwell's equations

a new vector quantity $\boldsymbol{A}$, which we call *vector magnetic potential* having units volt-seconds per metre (V·s·m^{-1}). To uniquely define a vector, we must define both its divergence and its curl. Let's define the curl of the vector $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{4}$$

As mentioned, to uniquely define a vector, we must specify its divergence as well as its curl. For example, say $\boldsymbol{A}$ only has an x-component for simplicity. (i.e., $A_y = A_z = 0$). Then, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ gives

$$B_x = 0 \tag{5a}$$
$$B_y = \partial A_x / \partial z \tag{5b}$$
$$B_z = -\partial A_x / \partial y \tag{5c}$$

which provides no information on the possible variation of A_x with x. If we knew the divergence of $\boldsymbol{A}$, i.e.

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x}, \tag{6}$$

our dilemma would be resolved. Here we define the curl of $\boldsymbol{A}$; the divergence we will handle shortly.

According to the curl definition we have made, $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = 0$ and we have satisfied Maxwell's equations. Hence,

$$\boldsymbol{H} = \frac{1}{\mu} \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{7}$$

Let's contrast this to scalar electric potential (V) we learnt in electrostatics. It was a scalar function, related to electric field through

$$\boldsymbol{E} = -\boldsymbol{\nabla} V. \tag{8}$$

Now, static electric fields behave quite differently from dynamic ones: they possess *no circulation* ($\nabla \times \boldsymbol{E} = 0$ or $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$) and hence are *conservative*. We can summarize differences between scalar and vector potentials by saying:

- If $\nabla \cdot \boldsymbol{M} = 0$, there exists a vector $\boldsymbol{G}$ such that $\boldsymbol{M} = \nabla \times \boldsymbol{G}$.
  i.e., if we have a circulating, non-conservative field $\boldsymbol{M}$, the vector potential of that field is the *curl* of the field.
- If $\nabla \times \boldsymbol{M} = 0$, there exists a function f such that $\boldsymbol{M} = \nabla f$.
  i.e., if we have a conservative field $\boldsymbol{M}$, the scalar potential of that field is the *gradient* of the field.

So, we have

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A}. \tag{9}$$

From Maxwell's first curl equation, we have

$$\begin{aligned} &\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H} = -j\omega\mu\left(\tfrac{1}{\mu}\nabla \times \boldsymbol{A}\right) = -j\omega\nabla \times \boldsymbol{A} \\ &\nabla \times \boldsymbol{E} + \nabla \times j\omega\boldsymbol{A} = 0 \\ &\Rightarrow \nabla \times \underbrace{(\boldsymbol{E} + j\omega\boldsymbol{A})}_{\text{a conservative electric field}} = 0. \end{aligned} \tag{10}$$

We notice we have a conservative electric field present since its curl is zero. Therefore, let the scalar potential of this field by denoted by ϕ such that

$$\boldsymbol{E} + j\omega\boldsymbol{A} = -\nabla\phi, \tag{11}$$

where we use a different symbol ϕ to distinguish it from the scalar potential V from statics, which was defined using (10). Then,

$$\boldsymbol{E} = -\nabla\phi - j\omega\boldsymbol{A}. \tag{12}$$

Let's substitute (9) into Maxwell's second curl equation:

$$\begin{aligned} &\nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J} \\ &\nabla \times \left(\tfrac{1}{\mu}\nabla \times \boldsymbol{A}\right) = j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J} \\ &\tfrac{1}{\mu}\underbrace{\nabla \times \nabla \times \boldsymbol{A}}_{\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}} = j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J} \end{aligned} \tag{13}$$

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A} = j\omega\varepsilon\mu\boldsymbol{E} + \mu\boldsymbol{J}. \tag{14}$$

Substituting in Equation (14),

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A} = j\omega\varepsilon\mu(-\nabla\phi - j\omega\boldsymbol{A}) + \mu\boldsymbol{J}, \tag{15}$$

or

$$\nabla^2\boldsymbol{A} + \omega^2\mu\varepsilon\boldsymbol{A} - \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\phi) = -\mu\boldsymbol{J}. \tag{16}$$

Previously, we defined $\nabla \times \boldsymbol{A}$. Now we need to define the divergence of $\boldsymbol{A}$, or $\nabla \cdot \boldsymbol{A}$, which we are at liberty to do. A convenient choice would be:

$$\nabla \cdot \boldsymbol{A} = -j\omega\varepsilon\mu V, \tag{17}$$

since it cancels the term in parentheses in the previous equation. Such a choice is called the *Lorenz condition* or *Lorenz gauge*. Implementing this gives

$$\begin{aligned} \nabla^2 \boldsymbol{A} + \underbrace{\omega^2\mu\varepsilon}_{k^2} \boldsymbol{A} &= -\mu \boldsymbol{J} & (18) \\ \nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} &= -\mu \boldsymbol{J}, & (19) \end{aligned}$$

where $k = \omega\sqrt{\mu\epsilon}$. Look familiar? It is the *vector wave equation* and it achieves part of what we set out to do in the first place: to write an explicit relationship between $\boldsymbol{A}$ and $\boldsymbol{J}$. To complete our goal, knowing $\boldsymbol{A}$ we can then find $\boldsymbol{E}$ and $\boldsymbol{H}$. $\boldsymbol{H}$ is easy:

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A}. \tag{20}$$

$\boldsymbol{E}$ can be found using Equation (14), where V is found using the Lorenz condition:

$$\begin{aligned} \boldsymbol{E} &= -j\omega \boldsymbol{A} - \nabla V & (21) \\ &= -j\omega \boldsymbol{A} - \nabla \left(\frac{\nabla \cdot \boldsymbol{A}}{j\omega\mu\varepsilon} \right) & (22) \\ &= -j\omega \boldsymbol{A} + \frac{j \nabla (\nabla \cdot \boldsymbol{A})}{\omega\mu\varepsilon}. & (23) \end{aligned}$$

However, an easier and often more straightforward approach is to simply use Maxwell's equations to find $\boldsymbol{E}$ using

$$\boldsymbol{E} = \frac{1}{j\omega\epsilon} \nabla \times \boldsymbol{H}. \tag{24}$$

So, we can summarize the steps we took to get from the currents to the radiated fields as

$$\boldsymbol{J} \Rightarrow \boldsymbol{A} \Rightarrow \begin{cases} \boldsymbol{H} \Rightarrow \boldsymbol{E} & \text{using Maxwell's equations to relate } \boldsymbol{H}, \boldsymbol{E} \\ \boldsymbol{E} & \text{using vector potential directly.} \end{cases} \tag{25}$$

ASIDE: Note that the Lorenz condition allows us to find $\boldsymbol{E}$ without ever finding ϕ, the scalar potential. But if we needed it, we know that from Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon} \tag{26}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot (-j\omega \boldsymbol{A} - \boldsymbol{\nabla}\phi) &= \rho/\varepsilon & (27) \\ -j\omega \boldsymbol{\nabla} \cdot \boldsymbol{A} - \underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi}_{\nabla^2\phi} &= \rho/\varepsilon, & (28) \end{aligned}$$

and from the Lorenz condition,

$$\begin{aligned} -j\omega(-j\omega\mu\varepsilon\phi) - \nabla^2\phi &= \rho/\varepsilon & (29) \\ \nabla^2\phi + \omega^2\mu\varepsilon &= \rho/\varepsilon. & (30) \end{aligned}$$

The last equation is the nonhomogenous wave equation in terms of the potential ϕ. So, really, we could solve for $\boldsymbol{E}$ using either approach, but using vector potentials versus scalar potentials is less cumbersome.

The vector wave equation

$$\boldsymbol{\nabla}^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} \tag{31}$$

can be expanded as

$$\begin{aligned} \nabla^2 A_x + k^2 A_x &= -\mu J_x & (32) \\ \nabla^2 A_y + k^2 A_y &= -\mu J_y & (33) \\ \nabla^2 A_z + k^2 A_z &= -\mu J_z, & (34) \end{aligned}$$

where $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We begin by solving an important problem of determining the solution of these equations when there is a point current source at the origin. Let $\boldsymbol{J} = \delta(x)\delta(y)\delta(z)\,\hat{\boldsymbol{a}}$, where $\hat{\boldsymbol{a}}$ is a unit vector indicating the direction of the current flow. We see that the current only exists at the origin (i.e. spatially, it is a point), and it has some direction associated with the current flow. We do this to find the response of $\boldsymbol{A}$ to a point source of current, because an arbitrary distribution of current can be represented as a continuous collection of point sources. Since the system of equations is linear, we can use *superposition* to determine $\boldsymbol{A}$ by summing all the contributions.

For the point source problem, we could solve all three equations by decomposing the current into all three components and solving the three equations. We will denote the solution to the wave equation for a point source at the origin as ψ.

A simple approach is to first assume the current is oriented along one of the principle directions, and then extrapolate to the general case of a current flowing in an arbitrary direction. Let's use a z-directed point source ($\hat{\boldsymbol{a}} = \hat{\boldsymbol{z}}$):

$$J_x = J_y = 0, \quad J_z = \begin{cases} 1 & \text{at the origin} \\ 0 & \text{everywhere else.} \end{cases} \tag{35}$$

Under this condition, $A_z = A_y = 0$, since there is no term to drive these components of the equations, hence yielding trivial solutions to the scalar Helmholtz equations for those components. The wave equation then simply becomes

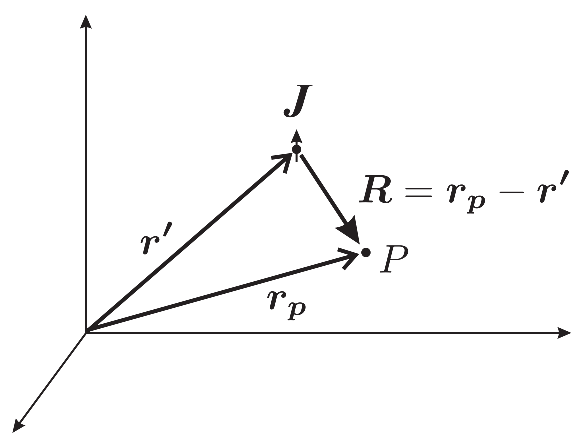
$$\nabla^2\psi + k^2\psi = -\mu\delta(x)\delta(y)\delta(z) \tag{36}$$

where ψ represents the solution to the wave equation under the specific condition of a point source. Due to the spherical nature of the problem, it is best to expand the Laplacian in spherical coordinates. Solving the resulting nonhomogenous differential equation yields a so-called *spherical wave* solution[1]:

$$\psi = \mu\frac{e^{-jkr}}{4\pi r}, \tag{37}$$

which we see is a wave propagating radially away from the origin, and also decaying with $1/r$ as it does so (another solution propagates towards the origin, growing without bound, and is not considered here as it is not physically meaningful).

Now let's extend this to a more general situation where the (z-directed) point source is located at some arbitrary point instead of the origin. Let this position be denoted by a position vector $\boldsymbol{r'}$. The the field (or observation) position have a position vector $\boldsymbol{r}_p$. Then, the distance from the source to the field point is R, according to the geometry in the following illustration.



The vector potential is then given by

$$\psi = \mu\frac{e^{-jkR}}{4\pi R}. \tag{38}$$

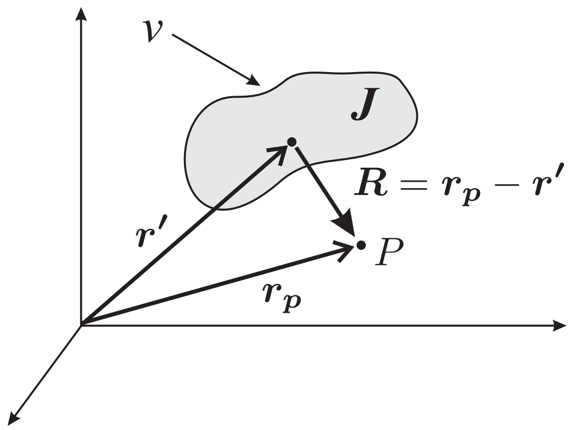
Now we consider an arbitrary z-directed current distribution $J_z(\boldsymbol{r'})$. The total response A_z is then the sum of all the individual point sources composing this distribution. If the current distribution is enclosed in a volume V, then the total vector potential is

$$A_z = \int_V \mu J_z(\boldsymbol{r'})\frac{e^{-jkR}}{4\pi R}dv'. \tag{39}$$

[1]This is derived in a separate note.

The equation is identical if there are other field components, allowing us to generalize the equation as

$$\boldsymbol{A} = \int_V \mu \boldsymbol{J}(\boldsymbol{r'}) \frac{e^{-jkR}}{4\pi R} dv'. \tag{40}$$



The steps in solving radiation problems can be re-summarized as: or,

1. Determine $\boldsymbol{A}$ from $\boldsymbol{J}$ using (42).
2. Find $\boldsymbol{H}$ from $\boldsymbol{A}$ using $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A}$.
3. Find $\boldsymbol{E}$ from $\boldsymbol{H}$ using $\boldsymbol{E} = \frac{1}{j\omega\epsilon}\boldsymbol{\nabla} \times \boldsymbol{H}$.