Friis' Formula and Effects

Friis' transmission formula in free space is

$$\frac{W_r}{W_t} = \frac{G_r G_t \lambda^2}{(4\pi R)^2}. \tag{1}$$

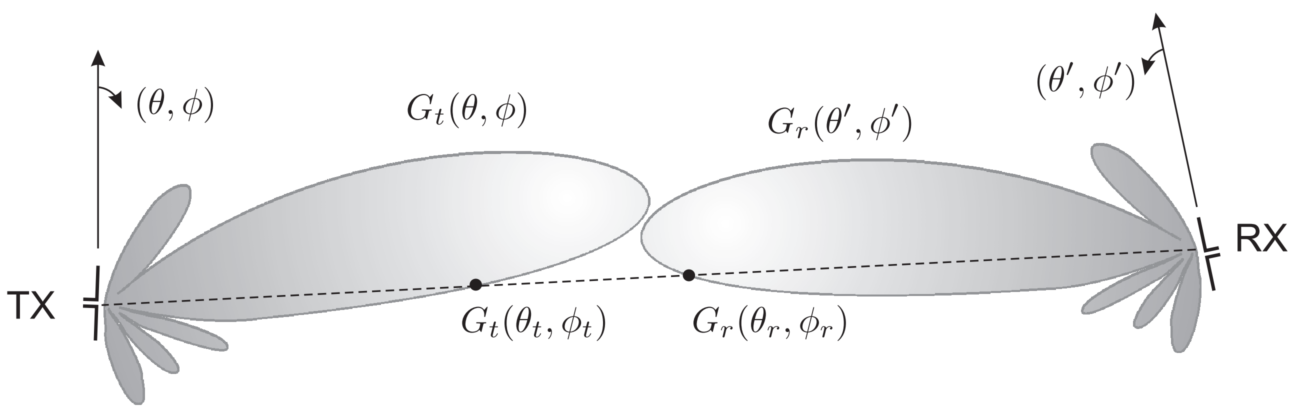
This equation assumes the following:

1. That the antennas are "pointed" at each other such that we obtain the maximum spatial response from each antenna (i.e., that G_r and G_t are the maximum gains associated with the antennas;
2. That the transmitter and receiver are perfectly impedance-matched to the antennas;
3. That the transmit and receive antennas are perfectly co-polarized (have the same polarization, and aligned for that polarization).

We will now modify Friis' formula to account for each of these effects.

1 Antenna Pointing

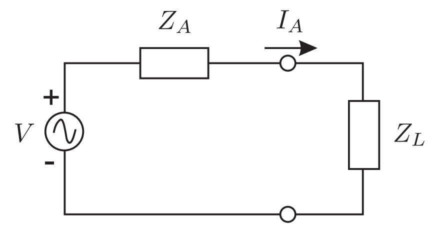
The effect of arbitrary transmitter/receiver antenna pointing can be easily addressed by making the gain terms in Friis' formula functions of pointing angle.



$$\frac{W_r}{W_t} = \frac{G_r(\theta_r, \phi_r) G_t(\theta_t, \phi_t) \lambda^2}{(4\pi R)^2} \tag{2}$$

2 Impedance Mismatch

Let's consider the model of the receive antenna presented previously:



The power delivered to the load resistor is

$$W_D = \frac{1}{2} I_A I_A^* R_L = \frac{1}{2} |I_A|^2 R_L, \tag{3}$$

where

$$I_A = \frac{V}{Z_A + Z_L} = \frac{V}{(R_A + jX_a) + (R_L + jX_L)}. \tag{4}$$

Therefore,

$$W_D = \frac{1}{2} \frac{|V|^2 R_L}{(R_A + R_L)^2 + (X_A + X_L)^2}. \tag{5}$$

The *maximum available power* from the source that is delivered to the load is produced when the load is conjugately matched to the source impedance; that is, $Z_L = Z_A^*$, yielding

$$W_{D,max} = \frac{1}{8} \frac{|V|^2}{R_A}. \tag{6}$$

If the load and antenna impedances are mismatched, the *impedance mismatch factor* represents the fraction of maximum available power that is actually delivered to the load. The remainder of the power is *reflected from the load*, essentially lost.

$$q_r = \frac{W_D}{W_{D,max}} = \frac{4 R_A R_L}{(R_A + R_L)^2 + (X_A + X_L)^2} \tag{7}$$

We can also express the impedance mismatch factor in terms of receiver reflection coefficient. Defining

$$\Gamma_r = \frac{Z_L - Z_A^*}{Z_L + Z_A}, \tag{8}$$

we can proceed as follows[1]:

$$|\Gamma_r| = \frac{(R_L + jX_L) - (R_A + jX_A)}{(R_L + jX_L) + (R_A + jX_A)} \tag{9}$$

$$|\Gamma_r|^2 = \frac{(R_L - R_A)^2 + (X_L + X_A)^2}{(R_L + R_A)^2 + (X_L + X_A)^2} \tag{10}$$

$$1 - |\Gamma_r|^2 = \frac{4 R_A R_L}{(R_A + R_L)^2 + (X_A + X_L)^2}. \tag{11}$$

[1] The conjugate in (8) follows from the fact that Z_A is complex, a situation not seen in ECE320/357.

Therefore,

$$q_r = 1 - |\Gamma_r|^2. \tag{12}$$

This situation also applies to the transmitter circuit, where an antenna of impedance Z_L could be seen as being driven by source with an impedance Z_A, as per the circuit diagram presented above. Hence, we can define a transmitter impedance mismatch factor q_t in the same manner:

$$q_t = 1 - |\Gamma_t|^2. \tag{13}$$

The modified Friis' formula becomes

$$\frac{W_r}{W_t} = \frac{G_r(\theta_r, \phi_r)G_t(\theta_t, \phi_t)\lambda^2 q_t q_r}{(4\pi R)^2} \tag{14}$$

3 Polarization Mismatch

The analysis of polarization mismatch between two antennas is facilitated through the vector effective length concept visited earlier in the course. Examining the equation for received terminal voltage, we see that polarization information about the incident wave is in $\boldsymbol{E}^i$ and polarization information about the receiving antenna is in $\boldsymbol{\ell}_{eff}$. Received power is proportional to $|V_r^{oc}|^2$; therefore, normalizing $|V_r^{oc}|^2$ as follows gives the fraction of power received when the incident field and antenna are potentially polarization mismatched. This factor is called the *polarization loss factor*:

$$p = \frac{|\boldsymbol{E}^i \cdot \boldsymbol{\ell}_{eff}|^2}{|\boldsymbol{E}^i|^2 |\boldsymbol{\ell}_{eff}|^2} \tag{15}$$

$$= \left| \frac{\boldsymbol{E}^i}{|\boldsymbol{E}^i|} \cdot \frac{\boldsymbol{\ell}_{eff}}{|\boldsymbol{\ell}_{eff}|} \right|^2 \tag{16}$$

$$= |\hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{\ell}}_{eff}|^2. \tag{17}$$

Here, $\hat{\boldsymbol{e}}^i$ is a complex unit vector describing the polarization of the incident wave (its *polarization vector*) and $\hat{\boldsymbol{\ell}}_{eff}$ describes the direction (polarization vector) of the vector effective length of the receiving antenna. We now make the following definitions:

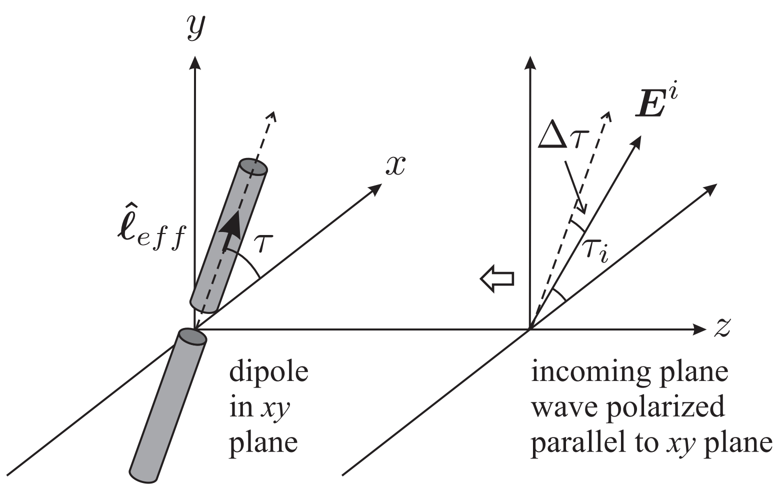
- If $|\hat{e}^i \cdot \hat{\boldsymbol{\ell}}_{eff}| = 1$, the incident wave and the antenna are said to be *co-polarized*. Maximum power transfer occurs between the incident wave and the antenna.
- If $|\hat{e}^i \cdot \hat{\boldsymbol{\ell}}_{eff}| = 0$, the incident wave and the antenna are said to be cross-polarized. No power transfer occurs between the incident wave and the antenna.

There are situations where each of these cases are desirable. Some applications use different polarizations for conveying independent signals; in this case, we want no cross-correlation between orthogonal polarization channels in the system. However, in other applications we may want the receiver to be insensitive to the polarization of the incoming wave.

Polarization Cases

Recall that the polarization vector of a wave is defined *when the antenna is in a transmitting mode*. Previously, when we discussed polarization, the $+z$ direction was assumed to be the direction of transmission, which we will assume here. Since the polarization vector (vector effective length) of an antenna is defined when it is transmitting, we need to need to consider an incident field approaching from the opposite direction in our analysis, which is the $-z$ direction.

Case 1: Receiving an LP wave with an LP antenna



In the transmitting antenna's coordinate system,

$$\hat{\boldsymbol{\ell}}_{eff} = \cos\tau\,\hat{\boldsymbol{x}} + \sin\tau\,\hat{\boldsymbol{y}} \tag{18}$$

Meanwhile, the incident electric field, expressed in the same coordinate system, has

$$\boldsymbol{E}^{inc} = E_0\hat{\boldsymbol{e}}^i e^{+jkz} \tag{19}$$

and

$$\hat{\boldsymbol{e}}^i = \cos\tau_i\,\hat{\boldsymbol{x}} + \sin\tau_i\,\hat{\boldsymbol{y}}, \tag{20}$$

where we note that the incident field is propagating in the $-z$ direction. Then,

$$\hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{\ell}}_{eff} = \cos\tau\cos\tau_i + \sin\tau\sin\tau_i = \cos(\tau_i - \tau) = \cos(\Delta\tau). \tag{21}$$

Therefore,

$$p = \cos^2(\Delta\tau). \tag{22}$$

Examples:

- $\Delta\tau = 0^\circ$, antenna aligned with incident field, $p = 1 = 0$ dB.
- $\Delta\tau = 45^\circ$, $p = \frac{1}{2} = -3$ dB: only have the power is intercepted by the receiving antenna.
- $\Delta\tau = 90^\circ$, antenna cross-polarized with incident field, $p = 0 = -\infty$ dB.

Case 2: Receiving an CP wave with a CP antenna

Question: How do we make a CP antenna? We know in general, for a wave travelling in the $+z$ direction,

$$\boldsymbol{E} = (E_1\,\hat{\boldsymbol{x}} + E_2\exp(j\delta)\,\hat{\boldsymbol{y}})e^{-jkz} = \boldsymbol{E}_0 e^{-jkz}. \tag{23}$$

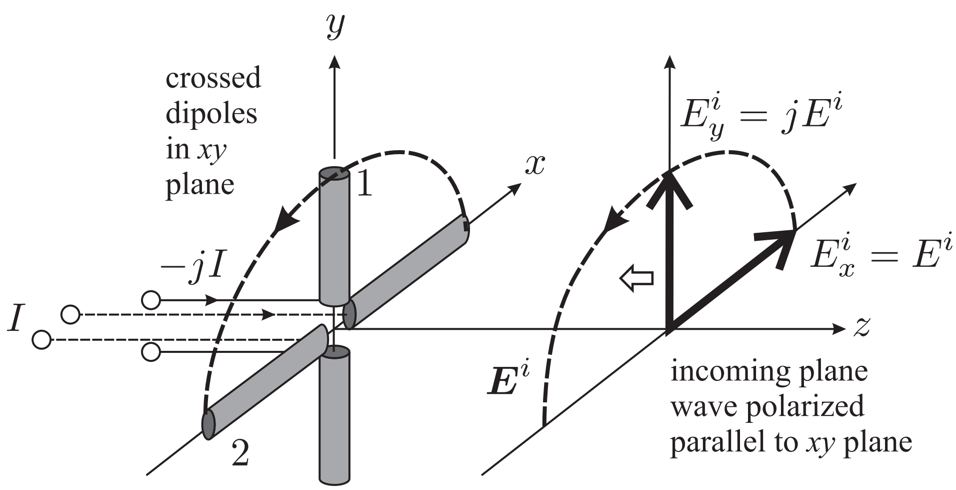
Recall that in CP antennas/waves, a $\pm 90^\circ$ phase shift is introduced between the x- and y-components of the radiated fields, i.e., $\delta = 90^\circ$ for LHCP and $\delta = -90^\circ$ for RHCP. The field magnitudes are also the same ($E_1 = E_2$). Therefore, for CP,

$$\boldsymbol{E}_0 = E_1\,\hat{\boldsymbol{x}} \pm jE_1\,\hat{\boldsymbol{y}} = \sqrt{2}E_1\hat{\boldsymbol{e}}_0, \tag{24}$$

where

$$\hat{\boldsymbol{e}}_0 = \frac{\hat{\boldsymbol{x}} \pm j\hat{\boldsymbol{y}}}{\sqrt{2}}. \tag{25}$$

From this we can see that we can make a CP wave with two linear cross-polarized antennas (e.g. dipoles) with a $\pm 90^\circ$ phase delay in one radiated field with respect to the other. In dipoles, for example, we know the radiated field is directly proportional to the drive current. Therefore, if we wanted to realize a CP antenna from two cross-polarized dipoles, we could do so be feeding one dipole directly and another via a $\lambda/4$ transmission line to create the 90° phase delay.



For this RHCP example, we can write

$$\hat{\boldsymbol{\ell}}_{eff} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} - j\,\hat{\boldsymbol{y}}). \tag{26}$$

We now consider an expression for the incident electric field. Let's say that the indident wave is RHCP. Then, in the coordinate system shown, we can see that

$$\hat{\boldsymbol{e}}^i = \frac{\hat{\boldsymbol{x}} + j\hat{\boldsymbol{y}}}{\sqrt{2}} \tag{27}$$

since the y-component must lead the x-component for an RHCP wave travelling in the $-z$ direction. The polarization loss factor between an incoming RHCP wave, and an RHCP antenna, is calculated as

$$\begin{aligned} p &= |\hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{\ell}}_{eff}|^2 & (28) \\ &= \left|\frac{1}{2}(\hat{\boldsymbol{x}} + j\,\hat{\boldsymbol{y}}) \cdot (\hat{\boldsymbol{x}} - j\,\hat{\boldsymbol{y}})\right|^2 & (29) \\ &= 1. & (30) \end{aligned}$$

Therefore, a RHCP-polarized antenna (one designed to produce RHCP during transmission) is *co-polarized* with an incident RHCP wave. Think of shaking hands with another person: you both use the same hand the grasp the other person's.

This also make sense when we think of how we made an RHCP antenna from two crossed dipoles and a $\lambda/4$ delay line. When receiving, the y-polarized signal enters the antenna and gets delayed by 90° with respect to the received x-polarized signal. But the y-polarized signal is already leading the x-polarized signal by 90°, so the delay line cancels this out and we get *constructive interference* when we add the two signals coming from the receive dipoles together.

For an LHCP wave incident upon an RHCP antenna, it is easy to show that

$$p = 0, \tag{31}$$

i.e., the antenna and wave are *cross-polarized*. Using the same argument above, we can see that this will lead to *destructive interference* when the two dipole received signals are added, and hence no power will be received.

Case 3: Receiving an CP wave with an LP antenna – or vice versa

We have shown that an LP antenna has

$$\hat{\boldsymbol{\ell}}_{eff} = \cos\tau\,\hat{\boldsymbol{x}} + \sin\tau\,\hat{\boldsymbol{y}}. \tag{32}$$

A CP wave has

$$\hat{\boldsymbol{e}}^i = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} \pm j\,\hat{\boldsymbol{y}}), \tag{33}$$

so

$$\begin{aligned} p &= |\hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{\ell}}_{eff}|^2 & (34) \\ &= \left|\frac{1}{2}(\hat{\boldsymbol{x}} \pm j\,\hat{\boldsymbol{y}}) \cdot (\cos\tau\,\hat{\boldsymbol{x}} + \sin\tau\,\hat{\boldsymbol{y}})\right|^2 & (35) \\ &= \left|\frac{1}{\sqrt{2}}(\cos\tau \pm j\sin\tau)\right|^2 & (36) \\ &= \frac{1}{2}(\cos^2\tau + \sin^2\tau) = \frac{1}{2}. & (37) \end{aligned}$$

So a LP antenna will receiving only have the power available from a CP wave. If we have a CP antenna receiving an LP wave, it is easy to prove that $p = 1/2$ also, as we might guess from a reciprocity argument.

In general we can define the unit vectors in terms of the polarization ellipse discussed earlier in the course, in the receiving antenna's coordinate system:

$$\begin{aligned} \hat{\boldsymbol{\ell}}_{eff} &= \cos\gamma\,\hat{\boldsymbol{x}} + \sin\gamma e^{j\delta}\,\hat{\boldsymbol{y}} & (38) \\ \hat{\boldsymbol{e}}^i &= \cos\gamma_i\,\hat{\boldsymbol{x}} + \sin\gamma_i e^{-j\delta_i}\,\hat{\boldsymbol{y}}. & (39) \end{aligned}$$

Note that there is not a sign error in the exponential term of (39): we need this sign reversal because the incident wave is traveling in the $-z$ in our coordinate system. This leads to

$$p = \left|\cos\gamma\cos\gamma_i + \sin\gamma\sin\gamma_i e^{j(\delta-\delta_i)}\right|^2. \tag{40}$$

This result is highly useful when working out polarization mismatch factors for cases we have not shown here (e.g. elliptical polarization).

Friis' transmission formula, which accounts for all of the effects we have discussed, is

$$\frac{W_r}{W_t} = \frac{G_r(\theta_r,\phi_r)G_t(\theta_t,\phi_t)\lambda^2 p q_t q_r}{(4\pi R)^2} \tag{41}$$

or

$$\frac{W_r}{W_t} = \frac{G_r(\theta_r,\phi_r)G_t(\theta_t,\phi_t)\lambda^2(1-|\Gamma_t|^2)(1-|\Gamma_r|^2|\hat{\boldsymbol{e}}^i\cdot\hat{\boldsymbol{\ell}}_{eff}|^2}{(4\pi R)^2}. \tag{42}$$

In dB form,

$$\begin{aligned} W_r|_{dBW/dBm} - W_t|_{dBW/dBm} &= G_r(\theta_r,\phi_r)|_{dB} + G_t(\theta_t,\phi_t)|_{dB} + 10\log_{10}(1-|\Gamma_t|^2) & (43) \\ &\quad +10\log_{10}(1-|\Gamma_r|^2) + 20\log_{10}|\hat{\boldsymbol{e}}^i\cdot\hat{\boldsymbol{\ell}}_{eff}| - 10\log_{10}(FSL) \end{aligned}$$

where $FSL = (4\pi R/\lambda)^2$. The "accounting" of loss factors is now starting to be come very apparent.