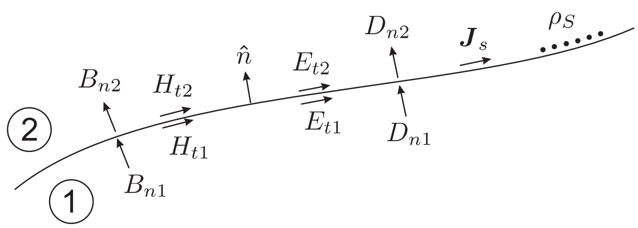
Reflection/Refraction

1 Boundary Conditions

Interfaces between different media imposed special boundary conditions on Maxwell's equations. It is important to understand what restrictions are placed on the electric and magnetic fields at a media interfance, since reflection and refraction of radio waves is described in terms of these boundary conditions.

To do this, we explicitly examine the *normal* and *tangential* components of the fields at a media interface. An example of an interface between two media, labelled as media 1 and 2, is illustrated below.

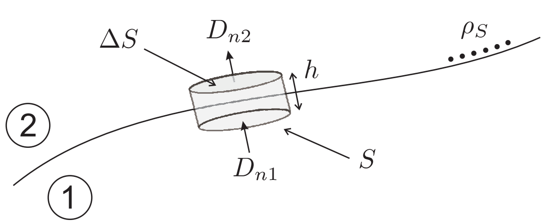


In this figure, we see electric and magnetic field density components, as well as a surface current density and a surface charge density denoted as $\boldsymbol{J}_s$ and ρ_s, respectively. The unit vector $\hat{n}$ is normal to the media interface and points toward medium 1.

Let's look at the normal components first, starting with the electric field density. The electric field density $\boldsymbol{D}$ is described by Gauss' Law:

$$\oint_S \boldsymbol{D} \cdot d\vec{S}' = \int_v \rho_v dv' \tag{1}$$

To evaluate the left hand side of the equation, let's use a small cylinder shown in the drawing below as the surface of integration.



The surfaces that contribute to the surface integral are the top of the cylinder, the bottom of the cylinder, and the side of the cylinder. If we shrink the cylinder so that it's height h tends to zero, then the dot products evaluated along the top and bottom of cylinder dominate the surface integral, and we have

$$\begin{aligned} \Delta S D_{n2} - \Delta S D_{n1} &= \Delta S \rho_s \\ \hat{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) &= \rho_s. \end{aligned} \tag{2}$$

This is the boundary condition for the normal component of the electric field density.

Similarly, due to the divergence-free nature of $\boldsymbol{B}$, we obtain

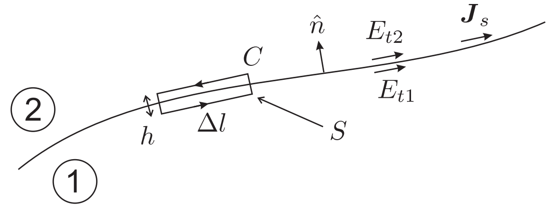
$$\begin{aligned}\oint_S \boldsymbol{B} \cdot d\vec{S}' &= 0 \\ \hat{n} \cdot \boldsymbol{B}_1 &= \hat{n} \cdot \boldsymbol{B}_2, \end{aligned} \tag{3}$$

which is the boundary condition for the normal component of the magnetic field density.

Now, for the tangential electric field, we use Faraday's Law, which states that

$$\oint_C \boldsymbol{E} \cdot d\vec{\ell} = -j\omega \int_S \boldsymbol{B} \cdot d\vec{S}'. \tag{4}$$

For the contour of integration, let's use a rectangular contour as illustrated below.



Like the previous case, we will consider the situation where the sides h of the rectangle are shrunk so that they are infinitisemally long, so that only the dot products along the length of the rectanglar are significant on the left hand side. Furthermore, on the right hand side, the area of the rectangle ΔS also tends to zero as $h \to 0$, yielding

$$E_{t1} = E_{t2} \tag{5}$$

or,

$$\hat{n} \times \boldsymbol{E}_1 = \hat{n} \times \boldsymbol{E}_2. \tag{6}$$

For the magnetic field, we use Ampere's Law:

$$\oint_C \boldsymbol{H} \cdot d\vec{\ell} = \int_S \boldsymbol{J} \cdot d\vec{S}'. \tag{7}$$

Again, as h tends to zero, only the tangential components of $\boldsymbol{H}$ make a significant contribution to the contour integral, while the right hand side reduces to the enclosed current *density*, J_s.

$$H_{t2} - H_{t1} = J_s, \tag{8}$$

or, equivalently,

$$\hat{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s. \tag{9}$$

Now, let's consider what happens when we consider specific cases for media interfaces. We will consider two: the interface between to dissimilar, dielectric media, and the interfance between a dielectric medium and a perfect electrical conductor. Both are extremely practical cases that we will deal with again later when we examine the reflection and refraction of radio waves at media interfaces.

1.1 Case 1: Dielectric Interface

At an interface between to dielectric media, $\rho_s = 0$, assuming the dielectrc is free of unbound charges. As well, dielectrics do not support current flow, and hence $\boldsymbol{J}_s = 0$. Subsitituting these conditions into Equations (2), (3), (6), and (9), we obtain the following:

$$\begin{aligned} \hat{n} \cdot \boldsymbol{D}_2 &= \hat{n} \cdot \boldsymbol{D}_1 & (10) \\ \hat{n} \cdot \boldsymbol{B}_2 &= \hat{n} \cdot \boldsymbol{B}_1 & (11) \\ \hat{n} \times \boldsymbol{E}_2 &= \hat{n} \times \boldsymbol{E}_1 & (12) \\ \hat{n} \times \boldsymbol{H}_2 &= \hat{n} \times \boldsymbol{H}_1. & (13) \end{aligned}$$

Hence we can make the following observations:

1. Normal components of $\boldsymbol{D}$ and $\boldsymbol{B}$ are continuous across a dielectric interface
2. Tangential components of $\boldsymbol{E}$, $\boldsymbol{H}$ are continuous across a dielectric surface

1.2 Case 2: Interface with Perfect Electrical Conductor

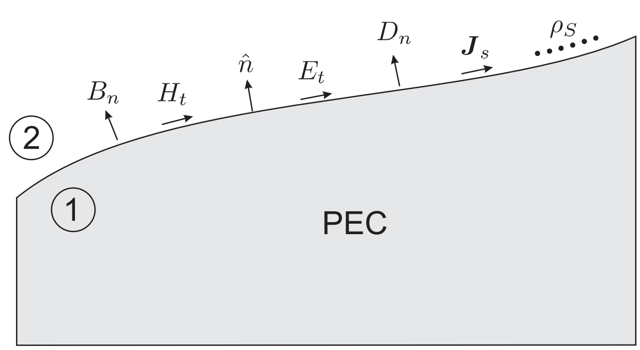
When we have a dielectric medium in contact with a perfect electrical conductor (PEC), a different set of field conditions are obtained. A PEC has infinite conductivity, i.e., $\sigma = \infty$. Hence, inside the PEC, from Ohm's Law,

$$\boldsymbol{J} = \sigma \boldsymbol{E} \Rightarrow \boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} \to 0 \text{ as } \sigma \to \infty. \quad (14)$$

Therefore, the electric fields are effectively "shorted out" in the PEC. The same is also true of the magnetic field (and magnetic field density):

$$\boldsymbol{B} = \frac{-1}{j\omega} \boldsymbol{\nabla} \times \boldsymbol{E} = 0 \Rightarrow \boldsymbol{B} = 0, \boldsymbol{H} = 0. \quad (15)$$

The situtation is summarized in the illustration below. We notice that since a PEC can support both surface charge and current densities, i.e., $\rho_s, \boldsymbol{J}_s \neq 0$, which directly impacts Equations (9).



Since fields can only exist on the dielectric side of the interface, we can summarize the effect on the normal and tangential field components as follows:

$$\hat{n} \cdot \boldsymbol{D} = \rho_s \tag{16}$$

$$\hat{n} \cdot \boldsymbol{B} = 0 \tag{17}$$

$$\hat{n} \times \boldsymbol{E} = 0 \tag{18}$$

$$\hat{n} \times \boldsymbol{H} = \boldsymbol{J}_s. \tag{19}$$

2 Plane Wave Reflections at a Media Interface

We will consider two types of reflection scenarios: reflection at an interface between two dielectric media, and reflection at an interface with a perfect electrical conductor. In each case we will derive the *Fresnel reflection coefficient* Γ which is in general a function of wave polarization, angle of incidence, and frequency.

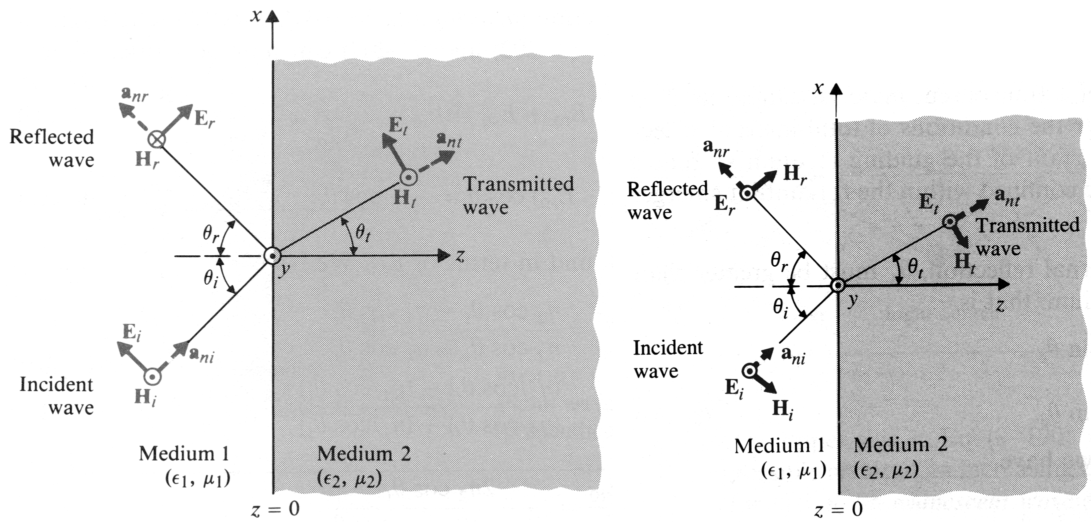
Consider the interface between two media. Medium 1 is assumed to have a permittivity ε_1 and permeability μ_1; likewise medium 2 has ε_2 and μ_2. They can be dielectric, conductors, or whatever you want. Now introduce a plane wave incident from medium 1. It consists of electric (E) and magnetic (H) fields. When it reaches the media interface, part of the wave is reflected, and part of it is transmitted. Let subscripts i, r, and t refer to the incident, reflected, and transmitted fields, respectively.

Consider the diagrams below. Rays indicating the direction of propagation are shown making angles θ_i, θ_r, and θ_t with the z-axis, or the axis normal to the interface. All rays are in the same plane, called the *plane of incidence*. The vectors shown in the diagram are for the electric field (which are obviously normal to the direction of propagation, right?), but bear in mind the magnetic field exists as well. Two specific cases for electric field polarization are shown (*reference directions* are shown). Case (a) shows the electric field polarized parallel to the plane of the page. Case (b) shows the case of the electric field polarized perpendicular to the page. Of course, a combination of these polarization is possible, but we will show the conditions for transmission and reflection reduce down to these two situations.

At a dielectric media interface, two phenomena can generally occur. There is a reflection back into the incident media, and there is transmission (in the form of *refraction*) into the second media. Conversely, if the second media is a perfect electrical conductor, then only reflection will occur. Let's consider the dielectric case first.

2.1 Reflection/Refraction at a Dielectric Interface

In the diagram, the waves can be polarized arbitrarily. We will consider two cases of polarization, since any wave can be decomposed into these two cases. The first is an electric field parallel to the plane containing all the wave propagation vectors (i.e., parallel to the page or the xz-plane). The other polarization is an electric field perpendicular to the xz-plane (parallel to the y-axis).



(a) Parallel polarization (b) Perpendicular polarization

Figure 1: Courtesy D. K. Cheng, "Field and Wave Electromagnetics", 2nd. Ed., Addison Wesley

2.1.1 Parallel Polarization

Consider the representation of the incident wave. The wavenumber in medium 1 is denoted as $k_1 = \omega\sqrt{\mu_1\varepsilon_1}$; since we have propagation at an angle θ_i to the z-axis, the vector wavenumber of the incident plane wave is represented as

$$\boldsymbol{k}_i = k_1 \sin\theta_i \,\hat{\boldsymbol{x}} + k_1 \cos\theta \,\hat{\boldsymbol{z}} \tag{20}$$

If the magnitude of the incident electric field is E_0, and the electric field vector lies in the xz-plane, we can write the electric and magnetic fields as:

$$\bar{E}_i = \boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\hat{\boldsymbol{r}}} = E_0(\hat{\boldsymbol{x}}\cos\theta_i - \hat{\boldsymbol{z}}\sin\theta_i)e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \tag{21}$$

$$\bar{H}_i = \boldsymbol{H}_0 e^{-j\boldsymbol{k}\cdot\hat{\boldsymbol{r}}} = \frac{E_0}{\eta_1}\hat{\boldsymbol{y}}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \tag{22}$$

where $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$. Similarly, the reflected and transmitted fields can be written as

$$\bar{E}_r = E_0\Gamma_{\parallel}(\hat{\boldsymbol{x}}\cos\theta_r + \hat{\boldsymbol{z}}\sin\theta_r)e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \tag{23}$$

$$\bar{H}_r = -\frac{E_0\Gamma_{\parallel}}{\eta_1}\hat{\boldsymbol{y}}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \tag{24}$$

$$\bar{E}_t = E_0T_{\parallel}(\hat{\boldsymbol{x}}\cos\theta_t - \hat{\boldsymbol{z}}\sin\theta_t)e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \tag{25}$$

$$\bar{H}_t = \frac{E_0T_{\parallel}}{\eta_2}\hat{\boldsymbol{y}}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \tag{26}$$

where the subscript $\parallel$ has been used to denote that the result was derived for an electric field polarized parallel to the plane containing the rays. $T_{\parallel}$ refers to the transmission coefficient from medium 1 into media 2; $k_2 = \omega\sqrt{\mu_2\varepsilon_2}$; and $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$.

We now enforce boundary conditions at the interface, which we derived earlier. Most notably, the tangential electric fields across the boundary must be continuous; likewise for the tangential magnetic fields. Therefore, at $z = 0$,

$$\cos\theta_i e^{-jk_1 x \sin\theta_i} + \Gamma_{\parallel} \cos\theta_r E^{-jk_1 x \sin\theta_r} = T_{\parallel} \cos\theta_t e^{-jk_2 x \sin\theta_t}; \tag{27}$$

$$\frac{1}{\eta_1} e^{-jk_1 x \sin\theta_i} - \frac{\Gamma_{\parallel}}{\eta_1} e^{-jk_1 x \sin\theta_r} = \frac{T_{\parallel}}{\eta_2} e^{-jk_2 x \sin\theta_t} \tag{28}$$

Note that this set of equations must hold everywhere along the media interface; that is, they must be invariant of x. Therefore, the arguments to the exponentials must all be equal (this is called the *phase matching condition*:

$$-jk_1 x \sin\theta_i = -jk_1 x \sin\theta_r = -jk_2 x \sin\theta_t \tag{29}$$

This results in two conditions:

$$\theta_i = \theta_r \tag{30}$$

$$k_1 \sin\theta_i = k_2 \sin\theta_t \tag{31}$$

which we immediate recognizes as Snell's law of reflection and refraction. We are perhaps more used to seeing Snell's law of refraction written as

$$\sqrt{\mu_1 \varepsilon_1} \sin\theta_i = \sqrt{\mu_2 \varepsilon_2} \sin\theta_t \tag{32}$$

which is usually applied in dielectric materials ($\mu_1 = \mu_2 = \mu_0$), so that

$$\sqrt{\varepsilon_1} \sin\theta_i = \sqrt{\varepsilon_2} \sin\theta_t. \tag{33}$$

$\sqrt{\mu\varepsilon}$ in general represents the *refractive index* of a medium.

The phase matching condition and law of reflection reduce Equations (27) and (28) to:

$$\cos\theta_i + \Gamma_{\parallel} \cos\theta_i = T_{\parallel} \cos\theta_t \tag{34}$$

$$\frac{1}{\eta_1} - \frac{\Gamma_{\parallel}}{\eta_1} = \frac{T_{\parallel}}{\eta_2} \tag{35}$$

Solving these equations simultaneously yields the following expressions for $\Gamma_{\parallel}$ and $T_{\parallel}$:

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{36}$$

$$T_{\parallel} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{37}$$

Observe that there is a special angle of incidence where $\Gamma_{\parallel}$ equals zero; that is, there is 100% transmission from medium 1 into medium 2. This occurs at an incidence angle equal to the

Brewster angle, which can be found by setting the numerator of the expression for $\Gamma_\|$ to zero:

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{b\|} \quad (38)$$

$$\frac{\eta_2}{\eta_1}\sqrt{1-\sin^2\theta_t} = \sqrt{1-\sin^2\theta_{b\|}} \quad (39)$$

$$\frac{\eta_2^2}{\eta_1^2}\left(1-\frac{k_1^2}{k_2^2}\sin^2\theta_{b\|}\right) = 1-\sin^2\theta_{b\|} \quad (40)$$

$$\frac{\mu_2\varepsilon_1}{\varepsilon_2\mu_1} - \frac{\varepsilon_1^2}{\varepsilon_2^2}\sin^2\theta_{b\|} = 1-\sin^2\theta_{b\|} \quad (41)$$

$$\frac{\mu_2\varepsilon_1}{\varepsilon_2\mu_1} - 1 = \left(\frac{\varepsilon_1^2}{\varepsilon_2^2}-1\right)\sin^2\theta_{b\|} \quad (42)$$

In dielectric media, $\mu_1 = \mu_2 = \mu_0$, and

$$\frac{\varepsilon_1}{\varepsilon_2} - 1 = \left(\frac{\varepsilon_1^2}{\varepsilon_2^2}-1\right)\sin^2\theta_{b\|} \quad (43)$$

The Brewster angle is then found using

$$\sin\theta_{b\|} = \frac{1}{\sqrt{1+\varepsilon_1/\varepsilon_2}}, \quad (44)$$

or, re-arranging,

$$\theta_{b\|} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}. \quad (45)$$

2.1.2 Perpendicular Polarization

In this case the electric field is polarized in the y-direction, or perpendicular to the plane containing the incident, reflected, and refracted waves. The fields can be represented as:

$$\bar{E}_i = E_0\hat{\boldsymbol{y}}\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \quad (46)$$

$$\bar{H}_i = \frac{E_0}{\eta_1}(-\hat{\boldsymbol{x}}\,\cos\theta_i + \hat{\boldsymbol{z}}\,\sin\theta_i)e^{-jk_1(x\sin\theta_i + z\cos\theta_i} \quad (47)$$

$$\bar{E}_r = E_0\Gamma_\perp\hat{\boldsymbol{y}}\, E^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \quad (48)$$

$$\bar{H}_r = \frac{E_0\Gamma_\perp}{\eta_1}(\hat{\boldsymbol{x}}, \cos\theta_r + \hat{\boldsymbol{z}}, \sin\theta_r)e^{-jk_1(x\sin\theta_r - z\cos\theta_r} \quad (49)$$

$$\bar{E}_t = E_0T_\perp\hat{\boldsymbol{y}}\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \quad (50)$$

$$\bar{H}_t = \frac{E_0T_\perp}{\eta_2}(-\hat{\boldsymbol{x}}\,\cos\theta_t + \hat{\boldsymbol{z}}\,\sin\theta_t)e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \quad (51)$$

where we observe the $\perp$ subscript used in the reflection/transmission coefficients in this case. Applying the boundary conditions and phase matching condition at the interface yields

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \quad (52)$$

$$T_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \quad (53)$$

Note that Snell's law governing the angle of refraction and refraction is unchanged for perpendicular polarization.

In dielectric media, for perpendicularly-polarized incident electric field, there is no Brewster angle as in the case for parallel polarization. This would require

$$\eta_2 \cos\theta_{b\perp} = \eta_1 \cos\theta_t \tag{54}$$

which, after applying Snell's law of refraction, would require

$$k_2^2(\eta_2^2 - \eta_1^2) = (k_2^2\eta_2^2 - k_1^2\eta_1^2)\sin^2\theta_{b\perp} \tag{55}$$

or

$$\sin\theta_{b\perp} = \sqrt{\frac{1 - \mu_1\varepsilon_2/\mu_2\varepsilon_1}{1 - \mu_1/\mu_2}}. \tag{56}$$

Since dielectrics have $\mu_1 = \mu_2 = \mu_0$, $k_2^2\eta_2^2 - k_1^2\eta_1^2 = 0$ which makes the equation above impossible to solve.

2.2 Reflection at Perfect Electrical Conductors

As you may recall, perfect electrical conductors have an reflection coefficient of $\Gamma = -1$. This is readily pr oven for either polarization case. For a perfect electrical conductor ($\sigma \to \infty$), the intrinsic impedance of medium 2 becomes

$$\lim_{\sigma\to\infty} \eta_2 = \lim_{\sigma\to\infty} \sqrt{\frac{j\omega\mu}{\sigma}} = 0. \tag{57}$$

Examining the expressions for reflection coefficient, we find that

$$\Gamma_\parallel = \Gamma_\perp = \Gamma = -1. \tag{58}$$

2.3 Grazing Angles at a Dielectric Interface

An important observation can be made about reflection/refraction at dielectric media interfaces with a very shallow angle of incidence. Basically, the incoming ray is nearly parallel to the surface (such that it "grazes" the surface). As the angle of incidence (θ_i) approaches $90°$, we notice the following:

$$\lim_{\theta_i\to 90°} \Gamma_\parallel = \frac{\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t} = 1 \tag{59}$$

$$\lim_{\theta_i\to 0°} \Gamma_\perp = \frac{-\eta_1 \cos\theta_t}{\eta_1 \cos\theta_t} = -1 \tag{60}$$

Interestingly, the magnitude of the reflection coefficients is 1 for grazing angles. Therefore, we can conclude that the magnitude of the reflections from a dielectric interface (even if the dielectrics are perfect) are the same as those that would have been produced by replacing the second dielectric with a perfect electrical conductor.

This is a useful result because often, the antenna is only a short distance above the ground, but the link distance is very large. Hence, grazing incident angles are produced and you get very good reflection from the surface of the earth, regardless of its electrical properties. It is important to note the subtle difference from the PEC case, however: the sign of the parallel reflection coefficient is opposite to that obtained for the PEC case. We will see this has implications on how we treat antennas over the earth, which in general is not a good conductor.