Array Directivity

1 Broadside Array

For the case where $d \ll \lambda$, the normalized array factor is

$$\text{(AF)}_n \approx \frac{\sin \left( \frac{N}{2} kd \cos \theta \right)}{\frac{N}{2} kd \cos \theta}$$

(1)

and the corresponding radiation intensity for an array of isotropic elements is

$$U(\theta) = [(\text{AF})_n]^2.$$  

(2)

The average radiation intensity is

$$U_0 = \frac{P_{\text{rad}}}{4\pi} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \left[ \frac{\sin \left( \frac{N}{2} kd \cos \theta \right)}{\frac{N}{2} kd \cos \theta} \right]^2 \sin \theta d\theta.$$  

(3)

Employing the substitution

$$Z = \frac{N}{2} kd \cos \theta$$

(4a)

$$dZ = -\frac{N}{2} kd \sin \theta d\theta$$

(4b)

we can write

$$U_0 = \frac{1}{2} \cdot \frac{-2}{Nkd} \left\{ \frac{\sin Z}{Z} \right\}^2 dZ \approx \frac{1}{Nkd} \int_{-\infty}^{\infty} \left[ \frac{\sin Z}{Z} \right]^2 dZ.$$  

(5)

if $Nkd$ is large (the array is long). In this case

$$\int_{-\infty}^{\infty} \left[ \frac{\sin Z}{Z} \right]^2 dZ = \pi$$  

(6)

and

$$U_0 \approx \frac{\pi}{Nkd},$$  

(7)

$$D_0 = \frac{U_{\text{max}}}{U_0} = U_0^{-1} = \frac{Nkd}{\pi} = 2N\frac{d}{\lambda}.$$  

(8)

2 Ordinary Endfire Array

Here we consider the case where $\theta_0 = 0^\circ$ or $\alpha = -kd$. The AF can then be written

$$\text{(AF)}_n \approx \frac{\sin \left[ \frac{N}{2} kd (\cos \theta - 1) \right]}{\frac{N}{2} kd (\cos \theta - 1)}$$  

(9)
with corresponding average radiation intensity

\[ U_0 = \frac{1}{2} \int_0^\pi \left[ \sin \left( \frac{N}{2} kd (\cos \theta - 1) \right) \right]^2 \sin \theta \, d\theta. \]  

To evaluate this, we use a similar substitution as for the broadside case,

\[ Z = \frac{N}{2} kd (\cos \theta - 1) \]

\[ dZ = -\frac{N}{2} kd \sin \theta \, d\theta \]

so that

\[ U_0 = -\frac{1}{N kd} \int_0^{-Nkd} \left[ \frac{\sin Z}{Z} \right]^2 \, dZ \approx \frac{1}{N kd} \int_0^\infty \left[ \frac{\sin Z}{Z} \right]^2 \, dZ \]  

if \( Nkd \) is large. \( U_0 \) is now half the value as it was for the broadside case, so that

\[ D_0 \approx 4N \frac{d}{\lambda}. \]  

### 3 Hansen-Woodyard Endfire Array

The Hansen-Woodyard array was derived to have an average radiation intensity given by

\[ U_0 = \frac{1}{2} \left\{ \frac{q(k - p)}{\sin[q(k - p)]} \right\}^2 \int_0^\pi \left\{ \frac{\sin[q(k \cos \theta - p)]}{q(k \cos \theta - p)} \right\}^2 \sin \theta \, d\theta. \]  

This quantity is minimized when \( q(k - p) = -1.46 \). Under this condition,

\[ U_0 = \frac{0.871}{Nkd} \]  

and

\[ D_0 = \frac{Nkd}{0.871} = \frac{1}{0.554} \left( \frac{2Nkd}{\pi} \right) = 1.805 \left[ 4N \left( \frac{d}{\lambda} \right) \right] \]

which is 1.8 times the directivity of an ordinary endfire array.